ON SELF-SIMILAR SOLUTIONS OF A TYPE OF A PROPAGATING WAVE OF A QUASI-LINEAR EQUATION AND OF A SYSTEM OF EQUATIONS WHICH DESCRIBE THE FLOW OF WATER IN A SLOPING CHANNEL

(OB AVTOMODEL'NYKH RESHENIIAKH TIPA RASPROSTRANIAIUSHCHEISIA VOLNY ODNOGO KVAZILINEINOGO URAVNENIIA I SISTEMY URAVNENII, OPISYVAIUSHCHEI TECHENIIA VODY V NAKLONNOM KANALE)

PMM, Vol. 30, № 2, 1966, pp. 303-311

E. B. Bykhovskii
(Leningrad)

(Received January 18, 1965)

1. Let us consider the quasi-linear equation

$$u_t + uu_x = F(u) \tag{1.1}$$

We shall investigate its self-similar solutions $u(\xi)=u(x-\omega t)$ which are continuous or have discontinuities of first order, are single-valued and determinate for all ξ .

As is well-known (cf., for example, [1]), in the case of a discontinuity of first order, the limiting values u_+ and u_- on either side of the discontinuity must satisfy conditions of the Hugoniot type $\frac{1}{2}(u_- + u_+)=\omega$ and a stability condition $u_+<u_-$.

On the portions of the functions $u(\xi)$, which are smooth, we have from (1.1)

$$(u - \omega)\frac{du}{d\xi} = F(u), \qquad \frac{d\xi}{du} = \frac{u - \omega}{F(u)} \tag{1.2}$$

We shall assume that $F(u)$ is a smooth function and that on any finite interval of variation of u, the function $F(u)$ does not have more than a finite number of changes of sign. The set of integral curves of (1.2) consists of curves in the (ξ,u)-plane on which ξ is a single-valued function of u, and all the curves of this set are obtained from some kind of a translation parallel to the ξ-axis.

If $F(u)\neq 0$ for $u=\omega$, then the sign of $d\xi/du$ in (1.2) changes for $u=\omega$ and, hence, a single-valued smooth self-similar solution $u(\xi)$ which is determinate for all ξ does not exist.

However for certain configurations of the integral curves of (1.2), for example, for those shown in Fig.1, a solution $u(\xi)$ having a discontinuity of first order can exist. But in this case there are also no solutions which have less than two discontinuities of first order. As is easily seen, such solutions exist if and only if the derivative $du/d\xi>0$ on the integral curves of (1.2) in some neighborhood of the line $u=\omega$, i. e. if $F(u)$ has a change of sign at $u=\omega$ from minus to plus (type(—+)). Thus the following theorem holds.

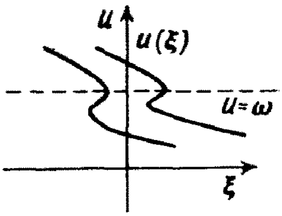

Fig. 1

T h e o r e m 1. 1. Let the smooth function $F(u)$ in Equation (1.1) have not more than a finite interval of variation of u. Let u_k ($k=0,1,2,\ldots$) be the changes of sign of $F(u)$ of the type (—+). Then self-similar solutions $u(\xi)=u(x-\omega t)$ are single-valued, determinate for all ξ, and have not less than two discontinuities of first order and exist only for $\omega_k=u_k$. There is a non-denumerable set of solutions indicated for each of these ω_k, even if no distinction is made between solutions obtained from one another by translations along ξ.

In Fig. 2 is shown a "simple periodic" solution which has one discontinuity in each period. If translations along the ξ-axis are disregarded, it is easily seen that the set of simple periodic solutions is a one-parameter family in which, for example, the height of the jump can be taken as the parameter.

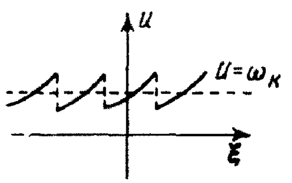

Fig.2

Each solution $u(x-u_0 t)$ can be considered as a solution (a generalized solution in the presence of discontinuities) of a Cauchy problem for Equation (1.1) with the initial conditions

$$u|_{t=0} = u(x).$$

As is well-known (cf., for example, [2]) if a finite rectangle $\{0\le t\le T, -x_0\le x\le x_0\}$ of the xt-plane is considered, these solutions in the norm L_1 with respect to x then depend continuously on the initial data. It is also well-known (cf. loc. cit.), that in the same norm they differ little from the solutions of the Cauchy problem for the equation

$$u_t + uu_x = F(u) + \mu u_{xx} \tag{1.3}$$

with the same initial data if μ is small.

Let us consider a self-similar solution $u(\xi)=u(x-u_0 t)$. Let us assume that Equation (1.1) describes a real phenomenon with small dissipation μu_{xx}, i. e. let $u(\xi)$ be a "truly observable" wave. In fact, since dissipation does exist, the truly observable wave $u(\xi)$ is then nothing other than the solution

$$u_\mu(\xi) = u_\mu(x - u_0 t)$$

of Equation (1.3) with small μ. Hence, the following definition is implied.

D e f i n i t i o n 1. 1. We shall call a self-similar solution $u(\xi)=u(x-u_0 t)$ self-similar stable in the interval $c<\xi<d$, if for any $\epsilon>0$, $\delta>0$ there can be found a $\mu_0(\epsilon,\delta,c,d)$ such that for any $\mu<\mu_0$ Equation (1.3) has a self-similar solution $u_\mu(\xi)=u_\mu(x-u_0 t)$, for which $u(\xi)$ satisfies the inequality $|u(\xi)-u_\mu(\xi)|<\varepsilon$ for $\xi\in(c,d)$ exterior to δ-neighborhoods of points of discontinuity.

The following theorem shows that the requirement of self-similar stability substantially restricts the class of possible self-similar solutions.

T h e o r e m 1. 2. In the interval u_0-a, u_0+a) let $F(u)$ have only one change of sign of the type $(-+)$ at the point $u=u_0$ and let $F(u_0+z)=-F(u_0-z)$ $(z<a)$.

Then if a solution $u(\xi)=u(x-u_0 t)$, whose values belong to the interval $(u_0-a,\ u_0+a)$, for $\xi\in(c,d)$ and which has just one discontinuity in the open interval $c<\xi<d$, is self-similar stable, it coincides with a simple periodic solution for $\xi\in(c,d)$.

P r o o f. First of all let us clarify the character of the solutions $u_\mu(\xi)$ for the assumptions which have been made regarding $F(u)$. For

$$u_\mu(\xi)=u_\mu(x-u_0 t)$$

Equation (1.3) gives the following ordinary differential equation:

$$(u_\mu-u_0)\frac{du_\mu}{d\xi}=F(u_\mu)+\mu\frac{d^2u_\mu}{d\xi^2} \tag{1.4}$$

The dynamic system of second order which corresponds to (1.4) has the form

$$\frac{dX}{d\xi}=Y,\qquad \mu\frac{dY}{d\xi}=XY-f(X) \tag{1.5}$$

Here

$$X=u_\mu-u_0,\qquad Y=\frac{d(u_\mu-u_0)}{d\xi}$$

$$f(X)=F(u_\mu),\quad f(z)=-f(-z)\quad (z<a)$$

The function $f(z)$ has a change of sign $(-+)$ at $z=0$. The system (1.5) has the point (0,0) as the center and its phase diagram is the strip $-a<X<a$ is presented in Fig. 3, where KL is a curve whose equation is $^-Y=X^{-1}f(X)$.

Consequently, all solutions of (1.4) with values from the interval $(u_0-a,\ u_0+a)$ are periodic. For brevity, let us designate the interval of variation of ξ for which the phase point passes through some arc of the cycle as the "time of passage" of this arc. The time of passage of the entire cycle is the period T which corresponds to $u_\mu(\xi)$. Over the period $0\le\xi\le T$ there is one interval of increase, the time of passage of that part of the cycle for which $Y>0$, and one interval of decrease, the time of passage of that part of the cycle for which $Y<0$.

Let us now assume the contrary statement of the theorem. Let the solution $u(\xi)$ which has just one discontinuity in (c,d) be self-similar stable in (c,d) but not coincident in this interval with any simple periodic solution. Then, in (c, d) there exists a point of discontinuity ξ_2 with the following properties: if (ξ_1, ξ_2) and (ξ_2, ξ_3) are intervals of smoothness of the function $u(\xi)$, which are adjacent from the left and from the right to the point ξ_2 (Fig.4) and which belong wholly to (c, d) and if (u_1, u_2^-) and (u_2^+, u_3) are the corresponding intervals of variation of $u(\xi)$, then either $u_2^- < u_3$ or $u_1 < u_2^+$ (Fig.4) (the notation $u_2^- = u(\xi_2 - 0)$ and $u_2^+ = u(\xi_2 + 0)$ has been introduced here).

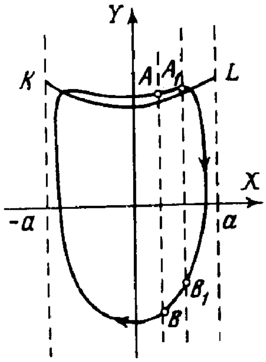

Fig.3

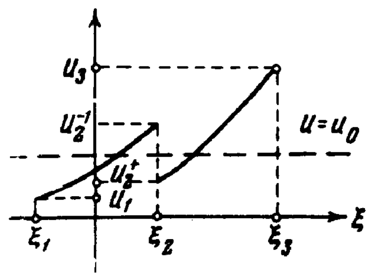

Fig.4

We shall examine, for example, the case $u_2^- < u_3$ and show that the assumption of self-similar stability of $u(\xi)$ leads to the absurd.

Let $u_\mu(\xi)$ be a solution which approximates $u(\xi)$ to within ϵ outside of the δ-neighborhoods of the discontinuities. Since the interval of variation of $u(\xi)$ for $\xi \in (c, d)$ is not less than (u_1, u_3), the interval 2α of the variation of $u_\mu(\xi)$ for $\xi \in (c, d)$ is then not less than $(u_1 + N\delta + \epsilon,\ u_3 - N\delta - \epsilon)$, where $N = \max |u'(\xi)|$ in the interval of smoothness of $u(\xi)$

In order that $u_\mu(\xi)$ approximate $u(\xi)$ to within ϵ exterior to the interval $(\xi_2 - \delta, \xi_2 + \delta)$, it is necessary, taking into consideration that $\epsilon < N\delta$, that the values of $u_\mu(\xi)$, which exceed u_2^- be concentrated in this interval, i. e. that the time of passage of the phase point along the arc AB of the cycle which lies to the right of the vertical line $X = u_2^- - u_0$ (Fig.3) must be less than 2δ.

On the other hand, let ξ_4 be a point of (ξ_2, ξ_3) at which $u(\xi_4) = u_2^-$. Then to approximate $u(\xi)$ in the interval $\xi_4 < \xi < \xi_3 - \delta$ it is necessary that the time of passage along the arc A_1B_1 which lies to the right of the straight line $X = u_2^- - u_0 + \epsilon$ be not less than $\xi_3 - \xi_4 - \delta$. Since the arc A_1B_1 is contained wholly within the arc AB, this leads to a contradiction for sufficiently small δ. The theorem is proved.

2. 1. Let us consider the system of equations which describe the flow of water in a sloping channel [3 and 4]

$$u_t + uu_x + gy_x = \Phi(u, y), \qquad y_t + [uy]_x = 0 \qquad (2.1)$$

$$\Phi(u, y) = g \tan\theta - \lambda u^m \left[\frac{ly}{l+2y}\right]^{-n} \operatorname{sign} u \qquad (2.2)$$

Here $u(x, t)$ is the velocity of the fluid particles, $y(x, t)$ is the depth of the flow, θ is the angle of inclination of the channel, l is its width, g is the acceleration due to gravity, $\lambda>0$ and m, $n>1$ are constants.

The system (2.1) can be written in the divergent form

$$(uy)_t + \left(u^2 y + g\frac{y^2}{2}\right)_x = y\Phi(u, y), \quad y_t + (uy)_x = 0 \qquad (2.3)$$

If the analogy to the system which describes the isotropic flow of a gas is drawn, then (2.3) is written in Eulerian coordinates [4] where the depth y plays the role of the gas density ρ. In analogy with gas dynamics let us now pass to Lagrangian coordinates [1]. In place of the independent variables x, t let us introduce new variables q, t' defined as

$$q = q(x, t), \quad t' = t, \quad dq = -y\,dx + uy\,dt \qquad (2.4)$$

The existence of the function $q(x, t)$ follows from the second equation of (2.3).

With this substitution

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - uy\frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial x} = y\frac{\partial}{\partial q}$$

Simultaneously with the substitution of the independent variables in (2.3) let us further introduce

$$v = y^{-1}, \quad p(v) = {}^1/_2 g v^{-2}$$

and in the final writing let us denote t' and q again by t and x, arriving at the following system

$$u_t + [p(v)]_x = F(u, v), \qquad v_t - u_x = 0 \qquad (2.5)$$

Here (*)

$$F(u, v) = \Phi(u, y) = a - \lambda u^m \left(v + \frac{2}{l}\right)^n \operatorname{sign} u \quad (a, \lambda > 0;\ m, n \geqslant 1) \qquad (2.6)$$

The system (2.5), which has a divergent form, may be regarded as a system of the "laws of conservation" which govern not only smooth flows, but also discontinuous flows (i. e., flows with "jumps"), if it is shown that the conditions on the jumps of the Hugoniot type which are implied in (2.5) coincide with physically realized conditions.

The conditions of Hugoniot type for (2.5), as is well-known [1], are

$$p(v_+) - p(v_-) = \omega(u_+ - u_-), \quad -(u_+ - u_-) = \omega(v_+ - v_-) \qquad (2.7)$$

*) A simple reference to the work in gas dynamics would be sufficient to transform the system (2.3) to Lagrangian coordinates; it is reproduced here since the term $\Phi(u, y)$ is absent in the gas dynamic equation.

Here ω is the Lagrangian velocity of diffusion of a discontinuity.

On the other hand, the real conditions on both sides of the jump in the flow have the form [3]

$$M(V_+ - V_-) = p_- - p_+, \qquad y_-V_- = y_+V_+ = M \tag{2.8}$$

Here M is the mass flow across the discontinuity, and V_- and V_+ are the velocities of the discontinuity relative to the particles on either side of it, i. e.

$V_- = U - u_-$, $V_+ = U - u_+$ (U is the velocity of discontinuity)

The identity of conditions (2.7) and (2.8) must be shown.

Taking the meaning of the Lagrangian velocity ω into account, we can affirm that $\omega = M$. Therefore, the first conditions of (2.7) and (2.8) are immediately coincident, and the second condition of (2.8) takes the form

$$U - u_- = v_-\omega, \qquad U - u_+ = v_+\omega$$

In addition, as is also the case in gas dynamics, we have the energy dissipation condition $v_+ > v_-$ for $\omega > 0$ and $v_+ < v_-$ for $\omega < 0$. We further note for later use that there follows from (2.7)

$$\frac{p(v_+) - p(v_-)}{v_+ - v_-} = -\omega^2 \tag{2.9}$$

Hence, considering the graph of $p(v)$ and taking the geometrical interpretation of ω^2 from (2.9) as a secant into account, we have the following: if that value of v_0 for which $p'(v_0) = -\omega^2$ is denoted by $v_0(\omega)$, then the value of v_0 always lies strictly between v_+ and v_-.

2. Let us consider the self-similar solution of (2.5)

$$u(\xi) = u(x - \omega t), \qquad v(\xi) = v(x - \omega t) \tag{2.10}$$

For such solutions (2.5) acquires the form

$$[p'(v) + \omega^2]\frac{dv}{d\xi} = F^*(v), \qquad u = -\omega v + C \tag{2.11}$$

Here

$$F^*(v) = F(-\omega v + C, v) \tag{2.12}$$

Let us consider the equation; for which ω (we shall call them **admissible** values) the existence of self-similar solutions of (2.5) having even one discontinuity (we shall call such solutions simply self-similar) is possible.

T h e o r e m 2. 1. 1°. All values of $\omega \le 0$ are inadmissible.

2°. The set of admissible $\omega > 0$ coincides with part of the semi-axis $\omega > 0$ for which the function

$$f(\omega) = m\omega - n\left(\frac{a}{\lambda}\right)^{\frac{1}{m}}\left(g^{\frac{1}{3}}\omega^{-\frac{2}{3}} + \frac{2}{l}\right)^{-\frac{n}{m}-1} \tag{2.13}$$

is negative.

In particular:

(a) For $2n/m < 1$ admissible ω exist and are bounded from above $\omega \le \omega_0$

$(m,\ n,\ l,\ a/\lambda)$.

(b) For $2n/m \geq 1$ the existence or nonexistence of admissible ω depends on the relations among the parameters, but in any case they are bounded from above just as in (a).

3°. For wide channels ($l=\infty$) the function $f(\omega)$ takes the form

$$f(\omega) = m\omega - n\left(\frac{a}{\lambda}\right)^{\frac{1}{m}} g^{-\frac{n}{3m}-\frac{1}{3}}\, \omega^{\frac{2n}{3m}+\frac{2}{3}} \tag{2.14}$$

(a) For $2n/m<1$ the set of admissible ω satisfies the inequality

$$0 < \omega < \omega_0 = n/m\left[\left(\frac{a}{\lambda}\right)^{\frac{1}{m}} g^{-\frac{n}{3m}-\frac{1}{3}}\right]^{\frac{3}{\alpha}} \tag{2.15}$$

$$(\alpha = 1 - 2n/m)$$

(b) For $2n/m>1$ the admissible values of ω satisfy the inequality $\omega_0<\omega$, where ω_0 is the same as in (a).

(c) For $2n/m=1$ (the Chezy case (Reference 3)) all values of ω are admissible if

$$2 - \left(\frac{a}{\lambda}\right)^{\frac{1}{m}} g^{-\frac{1}{2}} < 0 \tag{2.16}$$

and values of ω with opposite sign in the inequality are not at all admissible.

P r o o f. The question of the existence of self-similar solutions with specified values of ω which have even one discontinuity reduces to the following.

For an arbitrarily specified ω, let us find $v_0(\omega)$ such that $p'(v_0)+\omega^2=0$. Such a $v_0(\omega)>0$ can always be found and it is unique, namely $v_0(\omega) = g^{-1/3}\omega^{-2/3}$.

If now $C=C_0$ can be determined so that the properties of $F^*(v)$, defined by (2.12), guarantee the existence of integral curves $v=v(\xi)$ of equation (2.11) which monotonously increase for $\omega<0$ (Fig.5) and monotonically decrease for $\omega>0$ (Fig.6) in a horizontal strip of the (ξv)-plane, which includes the straight line $v=v_0(\omega)$ inside of it, then a self-similar solution with prescribed ω exists (*). In the converse case a self-similar solution does not exist.

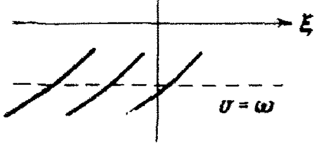

Fig.5

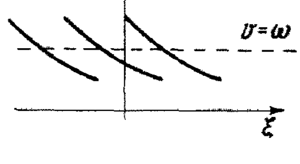

Fig.6

In fact, as has been shown above, $v_0(\omega)$ always lies between v_+ and v_- so that the range of values of the self-similar solutions $v(\xi)$ must include $v_0(\omega)$.

Further, for $\omega<0$ the dissipation condition has the form $v_+<v_-$ and a trend of the curves similar to that shown in Fig.5 is necessary (and sufficient) to be able to construct self-similar solutions with discontinuities.

*) In addition, a nondenumerable set of self-similar solutions also exists.

For $\omega > 0$ the dissipation condition is reversed and an analogous statement holds for Fig.6.

The behavior of the integral curves in the neighborhood of the straight line $v = v_0$ is, however, determined by the behavior of the function $F^*(v)$ at $v = v_0(\omega)$. Namely, since $p'(v) + \omega^2$ always has the change of sign $(- +)$ at v_0, then for the trend of the curves depicted in Fig.5 it is necessary and sufficient that $F^*(v)$ have a change of sign $(- +)$ at the point v_0 and for the trend of the curves in Fig.6 it is necessary and sufficient for $F^*(v)$ to have a change in sign $(+ -)$ at the point v_0

Thus, let ω be specified. For it $v_0(\omega) = g^{-1/3}\omega^{-2/3}$ can be found. We shall show that a $C = C_0(\omega)$ can always be found, and moreover it is unique, such that $F^*(v) = F(-\omega v + C_0, v)$ vanishes at $v = v_0$. Finding such a C reduces to solving the Equation

$$(-\omega v_0 + C)^m \operatorname{sign}(-\omega v_0 + C) = \frac{a}{\lambda}\left(v_0 + \frac{2}{l}\right)^{-n} \qquad (2.17)$$

for C.

From a graph of the left-hand side as a function of C it it is easily perceived that such a $C = C_0(\omega)$ can be found and is, moreover, unique.

Let us denote the function $F(-\omega v + C_0, v)$ by $F_\omega^*(v)$. It has the real root $v_0(\omega)$ (and there can still be others).

Let $\omega < 0$. The graph of the function

$$F_\omega^*(v) = a - \lambda(-\omega v + C_0)^m \left(v + \frac{2}{l}\right)^n \operatorname{sign}(-\omega v + C_0) \qquad (2.18)$$

is given in Fig.7. From it there is seen that $F_\omega^*(v)$ has a unique root which, consequently, is also v_0. But a change of sign $(+ -)$ occurs at this root which excludes the existence of a self-similar solution. Thus, part 1° of the theorem has been proved.

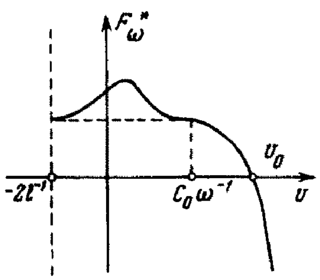

Fig.7

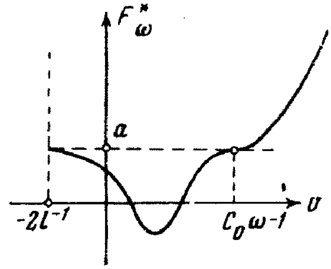

Fig.8

Let $\omega > 0$. The graph of the function $F_\omega^*(v)$ for $\omega > 0$ is given in Fig.8; at least one positive root, namely v_0, always exists.

The question consists of whether the necessary change of sign $(+ -)$ occurs at v_0 or, in other words, which of the two roots of the function $F_\omega^*(v)$ is v_0 (if both of them are positive, otherwise the question is resolved). The question is resolved by investigating the sign of the derivative $F_\omega^{*\prime}(v_0)$.

Taking into consideration the expressions for $F(u, v)$, $v_0(\omega)$ and the identity (2.17) and also the fact that all of the roots of $F_\omega^*(v)$ for $\omega > 0$, as can be seen from Fig.8, occur with $v < C_0/\omega$, i. e. with $-\omega v + C_0 > 0$, so that we can take sign $(-\omega v + C_0) = 1$ for the differentiation, and have

$$F_\omega^{*\prime}(v_0) = \lambda(-\omega v_0 + C_0)^{m-1}\left(v_0 + \frac{2}{l}\right)^n f(\omega) \qquad (2.19)$$

The change of sign $(+ -)$ necessary for the existence of self-similar solutions is guaranteed by the negativeness of $F_\omega^{*\prime}(v_0)$, i. e. by the negativeness of $f(\omega)$. We note that the case of $F_\omega^{*\prime}(v_0) = 0$ is inadmissible since in this case there is no change of sign of $F_\omega^*(v)$ at the point v_0 (Fig.8). This, by the way, shows that $\omega = 0$ is inadmissible.

To prove all of the points of parts 2° and 3° only an elementary investigation of the function $f(\omega)$ on the semi-axis $\omega>0$ remains to be carried out.

In the case (a) of point 2° it is easily seen that the second term in $f(\omega)$ is dominant for small ω so that $f(\omega)<0$ and the first term is dominant for large ω so that $f(\omega)\to+\infty$ as $\omega\to+\infty$.

This is valid for the case (a) of point 2°, but here it is further guaranteed that $f(\omega)$ has only one root and that it is easy to write an expression for it in terms of the parameters.

In the case (b) of point 1° the function $f(\omega)$ is positive for both small and large ω but, depending on the parameters, it can be negative for certain ω.

In the case (b) of point 2° the function $f(\omega)$ is positive for small ω and negative for large ω; moreover, a unique change of sign takes place at ω_0 which is indicated in the formulation of the theorem.

Finally, in the case (c) of point 2° we have

$$f(\omega)=\omega\left[m-n\left(\frac{a}{\lambda}\right)^{\frac{1}{m}} g^{-\frac{n}{3m}-\frac{1}{3}}\right] \tag{2.20}$$

Hence, this point is clearly confirmed. With this the proof of the theorem is completed.

N o t e 2. 1. The admissibility of certain $\omega>0$ means that the curve of $F_\omega^*(v)$ in Fig.8 has two positive roots, of which $v_0(\omega)$ is the smallest. Let ther be another root $v_1(\omega)$. At $v=v_1$ the function $F_\omega^*(v)$ has the change of sign $(-\ +)$; therefore, the integral curves have the form shown in Fig.9.

N o t e 2. 2. If the quantity $v_1(\omega)$ is taken as v_+ in the Hugoniot condition (2.9), then

$$v_{\min}(\omega)=[y_{\max}(\omega)]^{-1}$$

will play the role corresponding to v_- where $y_{\max}(\omega)$ is the maximum height of the wave possible for the given ω.

From Fig.8 it is seen that $v_1(\omega)<\omega^{-1}C_0(\omega)$. Taking into consideration that in the conditions (2.9) an increase of v_+ leads to a decrease of v_- and denoting by $\varphi(\omega)$ the quantity obtained from (2.9) as v_-, we have $v_{\min}(\omega)\geqslant\varphi(\omega)$ if $\omega^{-1}C_0(P\omega)$ is taken for v_+.

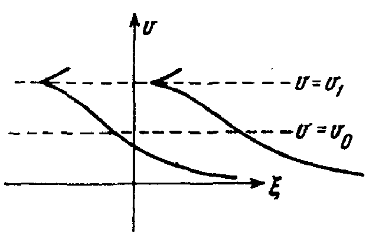

Fig.9

In the case (a) of points 2° and 3° admissible ω are bounded from above by a quantity $\omega_0(m,\ n,\ l,\ a/\lambda)$. (For point 3° there is no dependence on l). For the magnitude of $\inf v_{\min}(\omega)$ in the interval $0<\omega<\omega_0$, which is associated with the maximum height of a wave generally possible for the given parameters $m,\ n,\ l,\ a/\lambda$, we have the estimate

$$\min_{0\leqslant\omega\leqslant\omega_0}\varphi(\omega)\leqslant\inf_{0<\omega<\omega_0} v_{\min}(\omega)$$

The quantity standing here on the left can be easily written out explicitly as a function of the parameters $m,\ n,\ l,\ a/\lambda$ with the help of the equality (2.17) for C_0, the expression $v_0(\omega)=g^{-1/3}\omega^{-2/3}$ and the Hugoniot conditions (2.9).

In the case (b) of point 2° and , (b) and (c) of point 3°, where the admissible ω can be arbitrarily large, we have

$$v_{\min}(\omega)\leqslant v_0(\omega)\to 0 \qquad \text{for} \quad \omega\to+\infty$$

so that theoretically waves of arbitrarily large height are possible (clearly, for sufficiently large wave heights, the hydraulic equations (2.5) themselves become inapplicable.

N o t e 2. 3. From theorem (2.1) it is seen that the magnitude of the ratio m/n has an essential qualitative influence on the properties of the self-similar solutions, of which the very commonly used Chezy formula is an exceptional boundary case. Taking into consideration the empirical character of the quantities m and n, a question about the possibility of determining m/n to some degree from a qualitative examination of the behavior of a self-similar solution arises.

3. Analogously to the case of a single equation, the system (2.5) can be considered with a dissipation term

$$u_{+} + [p(v)]_x = \mu u_{xx} + F(u, v), \qquad v_t - u_x = 0 \tag{3.1}$$

and self-similar solutions of this system $u_\mu(x - \omega t)$, $v_\mu(x - \omega t)$ can be examined. In addition, the concept of the self-similar stability of the solutions can be introduced.

For $u_t(\xi)$, v_μ (ξ) the system (3.1) acquires the form

$$[p'(v) + \omega^2] \frac{dv}{d\xi} = F^*(v) - \omega\mu \frac{d^2v}{d\xi^2}, \qquad u(\xi) = -\omega v(\xi) + C \tag{3.2}$$

The dynamic system corresponding to the first equation of (3.1) is

$$\frac{dX}{d\xi} = Y, \qquad \omega\mu \frac{dY}{d\xi} = f(X) - \varphi(X)Y \tag{3.3}$$

Graphs of the functions $f(X)$ and $\varphi(X)$ are presented in Fig.10.

The functions $f(X)$ and $\varphi(X)$ in equation (3.3) are functions of a parameter ω, where ω has been chosen to be admissible in the sense of Theorem (2.1). The direction field for the system (3.3) is shown in Fig.11.

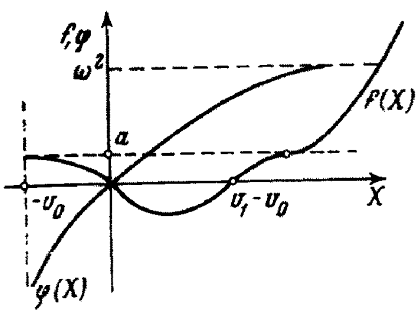

Fig.10

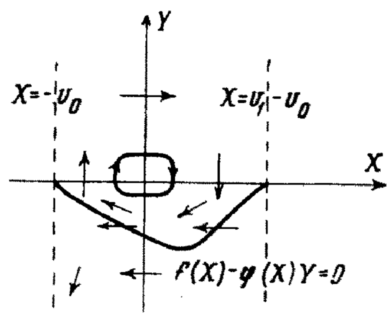

Fig.11

The point (0, 0) is the focus of center. The investigation of the phase diagram of (3.3) is a very interesting problem. If it turned out that the point (0, 0) is enclosed by a limit cycle, whose left-hand point does not approach infinitessimally close to the straight line $X = -v_0$ for all possible admissible ω, then this proclaims the existence of a maximum height of waves developed from an undisturbed flow even in the case for which the admissible ω can be arbitrarily large (cf. Note 22).

Let us approximate the functions $f(X)$ and $\varphi(X)$ in equation (3.3) by the linear functions $f(X) = l(X)$ and $\varphi(X) = kX$ in a small interval $[-h, h]$.

Then, in place of (3.3), we shall have the approximation

$$\frac{dX}{d\xi} = Y, \qquad \omega\mu \frac{dY}{d\xi} = lX - kXY \tag{3.4}$$

for the strip $-h \le X \le h$ of the phase plane.

The point (0, 0) will be the center for the system (3.4). The same reasoning as employed in Theorem 1.2 leads to the conclusion that only simple periodic solutions really exist (i. e. that there is self-similar stability).

In view of the fact that (3.4) approximates (3.3) only for small h, this conclusion, however, should for the present be considered valid only for waves of not large amplitude.

Its validity for waves of arbitrary amplitude is not excluded but is liable to further clarification.

BIBLIOGRAPHY

See next page

1. Gel'fand, I. M., Nekotorye zadachi teorii kvazilineinykh uravenii (Some problems in the theory of quasi-linear equations). Usp. mat. Nauk, Vol. 14, No. 2(86), 1959.

2. Oleinik, O. A., Razryvnye resheniia nelineinykh differentsial'nykh uravenii (Discontinuous solutions of nomlinear differential equations). Usp. mat. Nauk, Vol. 12, No. 3(75), 1957.

3. Dressler, R. F., Mathmatical Solution of the Problem of Roll-Waves in Inclined Open Channels. Communs. pure appl. Math., Vol. 2, 1949.

4. Dressler, R. F. and Pohle, F. V., Resistance Effects on Hydraulic Instability. Communs. pure and appl. Math., Vol. 6, No 1, 1953.